Toward the Emergence of Continuum Spacetime in Causal Set Theory

论因果集合理论中连续时空的涌现

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Abstract

摘要

In causal set theory (CST), the continuum spacetime manifold is postulated to be a low-energy approximation of a more fundamental discrete sub-structure of a locally finite partial ordered set, called as the causal set. In a collection of all n -element causal set Ω_n with large n , almost all causal sets are non-manifold-like, i.e., the one that cannot be approximated by a Lorentzian manifold of any dimension. These non-manifold-like causal sets have to be suppressed by the appropriate CST dynamics. This chapter contains a discussion primarily based on the work of Loomis and Carlip and that of Mathur, Singh and Surya on the suppression of a particular type non-manifold-like causal sets, which dominates Ω_n , in the causal set partition function (path-sum) by a simplified choice of causal set action. On these causal sets this choice is equivalent to the discrete Einstein-

在因果集理论 (CST) 中，连续时空流形被假定为一种更基础的离散子结构的低能近似，该子结构是局部有限的偏序集，被称为因果集。在所有 n 元因果集 Ω_n 的集合中，当 n 很大时，几乎所有因果集都不类似流形，即无法被任意维度的洛伦兹流形近似。这些不类似流形的因果集必须通过合适的因果集理论动力学抑制。本章基于卢米斯与卡利普、以及马图尔、辛格与苏里耶的工作展开讨论，内容为: 通过简化选择因果集作用量，在因果集配分函数 (路径和) 中抑制一类主导 Ω_n 的特殊非类流形因果集。对于这类因果集，该选择等价于离散爱因斯坦-

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Hilbert action, and hence, such causal sets are suppressed in the full causal set quantum partition function.

希尔伯特作用量，因此这类因果集会在完整因果集量子配分函数中被抑制。

Keywords

关键词

Causal sets - Path-sum - Kleitman-Rothschild orders - BDG action - Link action

因果集 - 路径和 - 克莱特曼-罗斯柴尔德序 - BDG 作用量 - 链接作用量

Introduction

引言

It is well established that our spacetime is a continuum manifold M equipped with a Lorentzian metric g . Therefore, in any quantum theory of gravity, it is required that we recover the manifold structure of the spacetime in appropriate limits, irrespective of the deep UV structure of the spacetime. In causal set approach to quantum gravity, the quantum spacetime has a structure of that of a locally finite partial ordered set, also called as the causal set $(C, \prec)$, where C is a set of spacetime elements, and $\prec$ denotes a partial order relation defined among the elements of C . This partial order $(\prec)$ corresponds to the causal order among the spacetime elements, i.e., for any two elements $e_i, e_j \in C, e_i \prec e_j$ denotes that e_j is in causal future of e_i . Given a spacetime manifold, a causal set can be obtained by picking elements from the manifold via a Poisson sampling process, also called here as ”sprinkling”, and assigning the partial order (-) from one element to the other if the latter is in the causal future of the former.

众所周知，我们的时空是一个配备洛伦兹度规的连续流形 M g 。因此，在任何量子引力理论中，无论时空的深层紫外结构如何，我们都需要在适当极限下恢复时空的流形结构。在因果集合量子引力方法中，量子时空具有局部有限偏序集的结构，也被称为因果集 $(C, \prec)$ ，其中 C 是时空元素的集合，$\prec$ 表示定义在 C 元素之间的偏序关系。这个偏序 $(\prec)$ 对应时空元素之间的因果序，即对任意两个元素 $e_i, e_j \in C, e_i \prec e_j$ 表示 e_j 位于 e_i 的因果未来。给定一个时空流形，可以通过泊松采样过程 (本文中也称为“洒播”) 从流形中选取元素，若一个元素在另一个元素的因果未来，则为两个元素赋予偏序关系，由此即可得到一个因果集。

Though every spacetime manifold can be thought of as an approximation to some causal set (with no nontrivial structure below the discreteness scale), not every causal set admits a manifold-like approximation in the sense discussed above. A causal set that cannot be approximated by a spacetime manifold is called as non-manifold-like causal set. It was shown by Kleitman and Rothschild in [1] that in a collection of causal sets with fixed but large number of elements, almost all causal sets are three-layered, which are obviously non-manifold-like. It was further shown in [2-4] that even if we remove all three-layered causal sets from the collection, the remainder will still be dominated by bilayer causal sets, followed by a hierarchy of layered causal sets. This means that if we randomly select a causal set from this collection, the probability of it being a manifold-like causal set is negligible. In other words, the manifold structure of the classical spacetime is not entropically favored in causal set theory (CST). The fact that we are living in a manifold-like universe then demands its explanation from the causal set dynamics. This explanation is crucial to the validity of the causal set approach to quantum gravity.

尽管任意时空流形都可以近似为某个因果集 (离散尺度以下不存在非平凡结构)，但并非所有因果集都存在上文所述的类流形近似。无法被时空流形近似的因果集被称为非类流形因果集。Kleitman 和 Rothschild 在文献 [1] 中证明，在元素数量固定且较大的因果集合中，几乎所有因果集都是三层结构，这类结构显然是非类流形的。进一步研究在文献 [2-4] 中表明，即使从该集合中移除所有三层因果集，剩余部分仍由双层因果集主导，之后是分层因果集的层级结构。这意味着如果我们从该集合中随机选取一个因果集，它是类流形因果集的概率可以忽略不计。换句话说，经典时空的流形结构在因果集理论 (CST) 中并不受熵青睐。我们生活在一个类流形宇宙中的这一事实，需要因果集动力学给出解释。这一解释对因果集量子引力方法的有效性至关重要。

This problem was partly addressed by Loomis and Carlip in [5], where they showed that the bilayer causal sets are suppressed in the causal set dynamics by the Benincasa-Dowker-Glaser (BDG) action for a suitable choice of action parameters. This result was partially generalized by Mathur, Singh and Surya in [6] to a family of collections of a particular type of layered causal sets called as quasi-layer orders (We will use the terms orders and causal sets interchangeably throughout this chapter.), all of which include the bilayer orders and the most dominant Kleitman-Rothschild (KR) orders, by showing their suppression in causal set dynamics by the link action for the same choice of action parameters. Carlip, Carlip and Surya later showed in [7] that the link action is approximately same as the BDG action for a three-layered order with large number of elements, which was then generalized to layered orders with all layers in [8]. Therefore, the analysis of [6] is applicable to the BDG action as well. We will discuss both the BDG and the link action in section ”Causal Set Partition Function”.

Loomis 和 Carlip 在文献 [5] 中部分解决了这个问题，他们证明，当对作用量参数选择合适取值时，双层因果集会在因果集动力学中被 Benincasa-Dowker-Glaser(BDG) 作用量压制。Mathur、Singh 和 Surya 在文献 [6] 中将这一结果部分推广到一类被称为拟层序的分层因果集集合 (本章中我们将“序”和“因果集”互换使用)，这类集合包含双层序和最占主导的 Kleitman-Rothschild(KR) 序；他们证明，当对作用量参数选择相同取值时，这类序都会在因果集动力学中被链接作用量压制。Carlip、Carlip 和 Surya 之后在文献 [7] 中证明，对于元素数量很大的三层序，链接作用量近似等同于 BDG 作用量，这一结论随后在文献 [8] 中被推广到所有层数的分层序。因此，文献 [6] 的分析也适用于 BDG 作用量。我们将在“因果集配分函数”一节讨论 BDG 作用量和链接作用量。

In this chapter, we discuss the work of Loomis and Carlip [5] and that of Mathur, Singh and Surya [6]. We start the chapter with a brief introduction to causal sets in section ”Causal Set Basics” followed by an introduction to a class of non-manifold-like causal sets called as layered causal sets in section ”Non-manifold-like Causal Sets”. We will be working with this class of non-manifold-like causal sets throughout this chapter. In section ”Causal Set Partition Function”, we discuss causal set actions and the partition function. In section ”Suppression of Layered Causal Sets”, we show that quasi-layer orders are suppressed in the causal set partition function for a suitable choice of causal set action parameters. We conclude the chapter in section ”Conclusions”.

本章我们将讨论 Loomis 和 Carlip[5] 以及 Mathur、Singh 和 Surya[6] 的相关工作。本章开篇我们将在“因果集基础”一节简要介绍因果集，随后在“非类流形因果集”一节介绍一类被称为分层因果集的非类流形因果集，我们将在整章都研究这类非类流形因果集。在“因果集配分函数”一节，我们讨论因果集作用量和配分函数。在“分层因果集的压制”一节，我们证明当合理选择因果集作用量参数时，拟层序会在因果集配分函数中被压制。我们在“结论”一节结束本章。

Causal Set Basics

因果集基础

This section contains a brief overview of causal sets and some basic definitions relevant to this chapter. For a detailed review on causal sets, the reader can refer to [9].

本节简要概述因果集及本章相关的基础定义。读者可参阅文献 [9] 获取因果集的详细综述。

Definition 1. A causal set $(C, \prec)$ is a locally finite partial order set. It is composed of a set C with partial order $\prec$ defined among its elements satisfying the following properties:

定义 1. 因果集 $(C, \prec)$ 是局部有限偏序集，由集合 C 与定义在其元素间的偏序关系 $\prec$ 构成，满足以下性质:

(i) Transitivity: For any $e_r, e_s, e_t \in C, e_r \prec e_s$ and $e_s \prec e_t \Rightarrow e_r \prec e_t$.

(i) 传递性: 对任意 $e_r, e_s, e_t \in C, e_r \prec e_s$ 和 $e_s \prec e_t \Rightarrow e_r \prec e_t$ 成立。

(ii) Asymmetry: There exists no $e_r, e_s \in C$ such that $e_r \prec e_s$ and $e_s \prec e_r$.

(ii) 非对称性: 不存在 $e_r, e_s \in C$ 满足 $e_r \prec e_s$ 且 $e_s \prec e_r$ 。

(iii) Local finiteness: For all $e_r, e_s \in C$, the set $\{e_t \in C \mid e_r \prec e_t \prec e_s\}$ is of finite cardinality.

(iii) 局部有限性: 对所有 $e_r, e_s \in C$ ， 集合 $\{e_t \in C \mid e_r \prec e_t \prec e_s\}$ 的基数有限。

Definition 2. The set $\{e_t \in C \mid e_r \prec e_t \prec e_s\}$ is called as the order interval of e_r and e_s and is denoted by $I[e_r, e_s]$.

定义 2. 集合 $\{e_t \in C \mid e_r \prec e_t \prec e_s\}$ 称为 e_r 和 e_s 的序区间，记为 $I[e_r, e_s]$ 。

Definition 3. A link $\prec *$ is a relation not implied by transitivity, i.e., $e_r \prec *e_s$ iff $e_r \prec e_s$ and $I[e_r, e_s] = \varnothing$

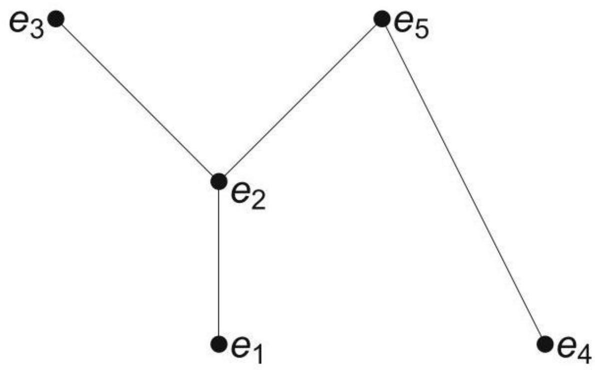
定义 3. 连接 $\prec *$ 是无法由传递性推导得到的关系，即 $e_r \prec *e_s$ 当且仅当 $e_r \prec e_s$ 且 $I[e_r, e_s] = \varnothing$

Definition 4. A minimal element e_r in C is the one with no preceding element, i.e., $\exists e_s \in C$ such that $e_s \prec e_r$.

定义 4. 极小元 e_r 在 C 中不存在前置元素，即不存在 $\exists e_s \in C$ 满足 $e_s \prec e_r$ 。

Fig. 1 The Hasse diagram of a 5-element causal set with e_1 and e_4 being the minimal element

图 1 含 5 个元素的因果集的哈塞图，其中 e_1 和 e_4 为极小元



A causal set can graphically be represented by its Hasse diagram, which only shows the link relations (the rest being implied by transitive closure), with the directed relation going from the bottom of the figure (past) to the top (future) as shown in Fig. 1.

因果集可通过哈塞图可视化表示，哈塞图仅展示连接关系，其余关系可由传递闭包推导得到；如图 1 所示，有向关系从图的底部 (过去) 指向顶部 (未来)。

Definition 5. A causal set $(C, \prec)$ is approximated by a continuum spacetime (M, g) iff there exists a countable subset M_C of M , obtained by sampling elements of M using a Poisson sampling process with a high enough density, and there exist a one-one onto map $\Phi : C \to M_C$ such that for any $e_r, e_s \in C, e_r \prec e_s \Leftrightarrow \Phi(e_s)$ is in the causal future of $\Phi(e_r)$.

定义 5. 因果集 $(C, \prec)$ 可由连续时空 (M, g) 近似，当且仅当存在 M 的可数子集 M_C (通过足够高密度的泊松采样过程从 M 中采样得到)，且存在满射一一映射 $\Phi : C \to M_C$ ，使得任意 $e_r, e_s \in C, e_r \prec e_s \Leftrightarrow \Phi(e_s)$ 都位于 $\Phi(e_r)$ 的因果未来中。

Definition 6. A causal set $(C, \prec)$ is manifold-like if there exists a continuum spacetime (M, g) such that $(C, \prec)$ is approximated by (M, g) .

定义 6. 若存在连续时空 (M, g) 使得因果集 $(C, \prec)$ 可由 (M, g) 近似，则称 $(C, \prec)$ 是流形状因果集。

Note that all the conditions on partial order $\prec$ in a causal set $(C, \prec)$ given by Definition 1, except ”local finiteness,” are satisfied by the causal ordering in the corresponding continuum spacetime (M, g) . A discussion on causal structure of a Lorentzian manifold can be found in [10,11].

注意，定义 1 给出的因果集 $(C, \prec)$ 中偏序 $\prec$ 的所有条件中，除“局部有限性”外，均满足对应连续时空 (M, g) 中的因果序关系。洛伦兹流形因果结构的相关讨论可参见文献 [10,11]。

The Poisson sampling process, also called as ”sprinkling”, is a method of sampling points from (M, g) in which the probability of picking n points in a given volume V is

泊松采样过程也称为“撒播”，是一种从 (M, g) 中采样点的方法，其中在给定体积 V 中选取 n 个点的概率为

$$P_V(n) = \frac{(\rho V)^n}{n!} e^{-\rho V}. \tag{1}$$

where ρ is called as the sprinkling density, which is determined by the fundamental cutoff. Over a large number of samplings, the average number of points in a region of volume $V, \langle N \rangle_V = \rho V$. This is how the number of elements is related to the volume of the spacetime region. Since the number of elements in a given region depends only on its volume, the Poisson sprinkling, unlike the lattice discretisation, provides a covariant discretization of the spacetime (M, g) (Fig. 2).

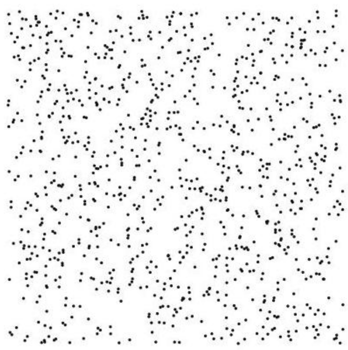
其中 ρ 为撒播密度，由基本截断决定。对大量采样而言，体积为 $V, \langle N \rangle_V = \rho V$ 的区域中的平均点数满足如下关系: 元素数与时空区域的体积直接相关。由于给定区域的元素数仅取决于区域体积，因此泊松撒播与晶格离散化不同，它能为时空 (M, g) 提供协变离散化 (图 2)。

It is shown in [1] that if we consider a sample space Ω_n of finite n -element causal sets, which are labelled over the set of n integers as in [1, 2, 4, 12] , almost none of the causal sets in Ω_n are manifold-like. The findings of [1] are even stronger, where

文献 [1] 表明，如果我们考虑有限 n 元素因果集的样本空间 Ω_n ——这些因果集像 [1, 2, 4, 12] 中那样被标记在 n 个整数构成的集合上——那么 Ω_n 中几乎没有类流形因果集。文献 [1] 的结论还可以进一步强化，即

Fig. 2 A 2d manifold-like causal set obtained from a square in a 2d spacetime by the process of Poisson-sprinkling

图 2 通过泊松撒播从二维时空的正方形中得到的二维类流形因果集



it is shown that almost all of the causal sets in Ω_n in the large n limit form a special class of three-layered orders, which we shall define in section "Non-manifold-like Causal Sets".

研究表明，在大 n 极限下，Ω_n 中几乎所有因果集都构成一类特殊的三层序，我们将在“非类流形因果集”一节给出定义。

Non-manifold-like Causal Sets

非流形型因果集

In this section we look at layered causal sets. They are non-manifold-like and almost entirely populate Ω_n with large n . We define different types of layers in a causal set and give several examples of them along the way. These definitions are taken directly from [6].

本节我们研究分层因果集。这类因果集属于非流形型，几乎完全由大型 n 填充 Ω_n 。我们定义因果集中不同类型的层，并在过程中给出多个实例。这些定义直接引自文献 [6]。

Definition 7. The level L_j of a causal set C , with $j = 1, 2, 3, \ldots k$, is the set of minimal elements that remain after deleting all elements in levels $L_m, m < j$. In particular, L_1 contains all the minimal elements of C [1] (Fig. 3).

定义 7。给定 $j = 1, 2, 3, \ldots k$ ，因果集 C 的层 L_j 是删除层 $L_m, m < j$ 中所有元素后剩余的极小元集合。特别地，L_1 包含 C 的所有极小元 [1](图 3)。

As is obvious from this definition, a level can be assigned to any causal set.

由该定义易知，任意因果集都可以定义层。

Definition 8. A Kleitman-Rothschild (KR) order has three levels L_1, L_2, L_3 satisfying the following properties:

定义 8。Kleitman-Rothschild(KR) 序具有三个层 L_1, L_2, L_3 ，满足以下性质:

1. $|L_1|, |L_3| = n/4 + o(n)$ and $|L_2| = n/2 + o(n)$.

1. $|L_1|, |L_3| = n/4 + o(n)$ 和 $|L_2| = n/2 + o(n)$ 。

2. $e_r \prec *e_s$ and $e_r \in L_j$ implies $e_s \in L_{j+1}$.

2. $e_r \prec *e_s$ 和 $e_r \in L_j$ 蕴含 $e_s \in L_{j+1}$ 。

3. Each element in a level L_j is connected to asymptotically half of the elements in L_{j-1} and half of the elements in L_{j+1} .

3. 层 L_j 中的每个元素，渐近地连接 L_{j-1} 中一半的元素与 L_{j+1} 中一半的元素。

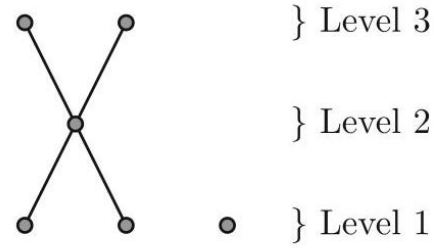
4. For all $e_r \in L_1$ and $e_s \in L_3, e_r \prec e_s$ (Fig. 4).

4. 对所有 $e_r \in L_1$ 和 $e_s \in L_3, e_r \prec e_s$ (图 4)。

Definition 8-4 is not explicitly stated in the original paper of Kleitman and Rothschild [1]. However since the dominant contribution comes from those orders satisfying all the conditions in Definition 8, this additional condition was imposed in [2, 4, 12] . It was later shown that in addition to the KR orders, there is a hierarchy of subdominant k -layer orders [2-4], which are defined as follows.

定义 8 的第 4 条没有在 Kleitman 和 Rothschild 的原始论文 [1] 中明确给出。但由于满足定义 8 中所有条件的序贡献占主导，该附加条件已在 [2, 4, 12] 中引入。后续研究表明，除 KR 序外，还存在次主导的 k 层序层级 [2-4]，定义如下。

Fig. 3 The Hasse diagram of a 6-element causal set with 3 levels



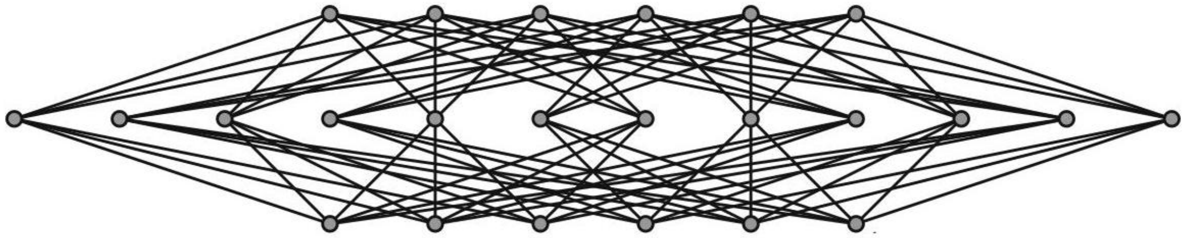


Fig. 4 An example of a 24-element causal set, which satisfies the KR property

图 4 满足 KR 性质的 24 元因果集示例

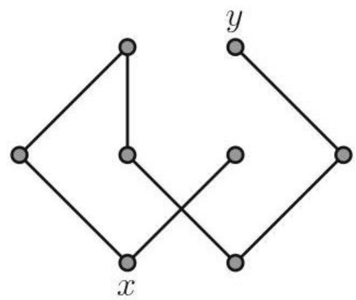


Fig. 5 An example of a causal set that is not a k -layer order for any k . While it is possible to assign a map $\zeta : c \rightarrow \mathbb{N}$ which satisfies Condition 1 of Definition 9, as shown in the figure, it cannot satisfy Condition 2, since element x with $\zeta(x) = 1$ and element y with $\zeta(y) = 3$ are not related

图 5 对任意 k 都不属于 k 层序的因果集示例。如图所示，虽然可以给它赋予满足定义 9 条件 1 的映射 $\zeta : c \rightarrow \mathbb{N}$ ，但它无法满足条件 2: 带有 $\zeta(x) = 1$ 的元素 x 和带有 $\zeta(y) = 3$ 的元素 y 之间不存在因果关系

Definition 9 ([2, 4, 12]). In a k-layer causal set $C \in \Omega_n$ it is possible to assign a k -layer-map $\zeta(e_r) \in \{1, 2, \ldots, k\}$ to each element $e_r \in C$, such that:

定义 9 ([2,4,12])。在 k 层因果集 $C \in \Omega_n$ 中，可以给每个元素 $e_r \in C$ 分配一个 k 层映射 $\zeta(e_r) \in \{1, 2, \ldots, k\}$ ，满足:

1. $e_r \prec e_s \Rightarrow \zeta(e_r) < \zeta(e_s)$.
2. $\zeta(e_s) > \zeta(e_r) + 1 \Rightarrow e_r \prec e_s$.

Let $\mathcal{D}_n^k$ denote the set of these orders.

设 $\mathcal{D}_n^k$ 为这些序的集合。

The class of k -layer orders is special and not every causal set is a k -layer order for any choice of k . A counter example of the k -layer order is shown in Fig. 5.

k 层序类是特殊的，并非任意因果集对任意选择的 k 都属于 k 层序。图 5 给出了一个 k 层序的反例。

Definition 10. A subset $\hat{C} \subset C$ is causally disconnected if there exists no relation between elements of $\hat{C}$ and its complement $\hat{C}^c$ in $C.\hat{C}$ is an irreducible causally disconnected subset of C if further, it contains no nontrivial causally disconnected proper subsets.

定义 10. 若 $\hat{C}$ 中的元素与其在 $C.\hat{C}$ 中的补集 $\hat{C}^c$ 之间不存在关系，则子集 $\hat{C} \subset C$ 是因果不连通的。若 $\hat{C} \subset C$ 进一步不包含非平凡的因果不连通真子集，则它是 C 的不可约因果不连通子集。

Fig. 6 An example of a causal set that is not a k -QL order for any k . Element x with $\eta(x) = 1$ and element y with $\eta(y) = 3$ are related by a link thus violating Condition 1 of Definition 11

图 6 一个对任意 k 都不是 k -QL 序的因果集示例。元素 x 与 $\eta(x) = 1$ 、元素 y 与 $\eta(y) = 3$ 通过链路相连，因此违反了定义 11 的条件 1

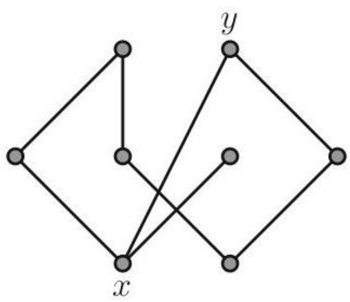
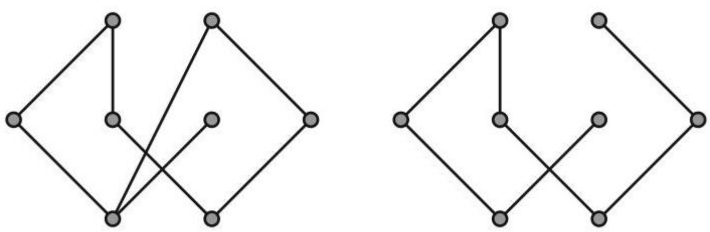


Fig. 7 The first causal set belongs to $\mathcal{D}_n^k$ but not to $\mathcal{Q}_n^k$, and the second to $\mathcal{Q}_n^k$ but not to $\mathcal{D}_n^k$

图 7 第一个因果集属于 $\mathcal{D}_n^k$ 但不属于 $\mathcal{Q}_n^k$ ，第二个因果集属于 $\mathcal{Q}_n^k$ 但不属于 $\mathcal{D}_n^k$



Definition 11 ([6]). In a k -quasi-layer (k -QL) order $C \in \Omega_n$, it is possible to assign a k -layer-map $\eta \in \{1, 2, \ldots, k\}$, such that:

定义 11(文献 [6])。在一个 k -拟层 (k -QL) 序 $C \in \Omega_n$ 中，可以分配一个 k -层映射 $\eta \in \{1, 2, \ldots, k\}$ ，满足:

1. For $e_r, e_s \in C$, if $e_r \prec *e_s$ then $\eta(e_s) = \eta(e_r) + 1$.

1. 对 $e_r, e_s \in C$ ，若 $e_r \prec *e_s$ 则 $\eta(e_s) = \eta(e_r) + 1$ 。

2. For every causally disconnected subset $\widehat{C} \subset C, \exists e_\alpha \in \widehat{C}$ such that $\eta(e_\alpha) = 1$.

2. 对任意满足 $\eta(e_\alpha) = 1$ 的因果不连通子集 $\widehat{C} \subset C, \exists e_\alpha \in \widehat{C}$ 。

We will refer to this k -layer-map assignment as quasi-layers (QL). Let $\mathcal{Q}_n^k$ denote the set of k -QL orders. Just like k -layer orders, not all causal sets are k -QL orders. A counterexample of the k -QL order is shown in Fig. 6

我们将这种 k -层映射分配称为拟层 (QL)。设 $\mathcal{Q}_n^k$ 为 k -QL 序的集合。和 k -层序一样，并非所有因果集都是 k -QL 序。图 6 给出了一个 k -QL 序的反例

It is shown in [6] that for a given labelled causal set the assignment of QLs, if any, is unique, i.e., every k -QL order is a unique labelled order in $\Omega_n.\mathcal{Q}_n^k$ is therefore a proper subset of Ω_n . It is evident that $\mathcal{D}_n^k \cap \mathcal{Q}_n^k \neq \varnothing$, but that one is not nested inside the other. This is because $\exists C \in \mathcal{D}_n^k$ in which there are links between nonconsecutive layers, and hence Definition 11-1 is not satisfied. Conversely, $\exists C \in \mathcal{Q}_n^k$ such that the elements in the $(i+2)$ th QL are not all related to those in the i th QL and hence, Definition 9-2 is not satisfied. Importantly, since the KR orders satisfy both Definitions 11-1 and 9-2, they lie in $\mathcal{D}_n^k \cap \mathcal{Q}_n^k$. We give examples of these differences in Fig. 7. Note also that typical manifold-like causal sets do not lie in either $\mathcal{Q}_n^k$ or $\mathcal{D}_n^k$.

文献 [6] 已证明，对于给定的标记因果集，拟层分配 (若存在) 是唯一的，即每个 k -QL 序在 $\Omega_n.\mathcal{Q}_n^k$ 中都是唯一的标记序，因此 Ω_n 是 $\Omega_n.\mathcal{Q}_n^k$ 的真子集。显然 $\mathcal{D}_n^k \cap \mathcal{Q}_n^k \neq \varnothing$ ，但二者不存在互相包含关系。这是因为存在 $\exists C \in \mathcal{D}_n^k$ ，其中非相邻层之间存在链路，因此不满足定义 11 的条件 1。反之，存在 $\exists C \in \mathcal{Q}_n^k$ ，满足第 $(i+2)$ 个拟层中的元素并非都与第 i 个拟层中的元素相关，因此不满足定义 9 的条件 2。重要的是，由于 KR 序同时满足定义 11-1 和定义 9-2，因此它们属于 $\mathcal{D}_n^k \cap \mathcal{Q}_n^k$ 。我们在图 7 中给出了这些差异的示例。还需注意，典型的流形状因果集既不属于 $\mathcal{Q}_n^k$ 也不属于 $\mathcal{D}_n^k$ 。

Another important class of causal sets is that of bilayer (2-layer) orders, in which all relations are links. Every $C \in \mathcal{Q}_n^2$ is a bilayer order. Conversely, to any bilayer order C , we can assign the QL $\eta = 1$ for all minimal elements and $\eta = 2$ otherwise. This satisfies Definitions 11-1 and 11-2 and hence, $C \in \mathcal{Q}_n^2$. In other words,2-layer orders and 2-QL orders form the same class of causal sets.

另一类重要的因果集是双层 (2 层) 序，这类因果集中所有关系都是链路。每个 $C \in \mathcal{Q}_n^2$ 都是双层序。反之，对任意双层序 C，我们可以为所有极小元分配 QL $\eta = 1$，为其他元分配 QL $\eta = 2$。这满足定义 11-1 和 11-2，因此属于 $C \in \mathcal{Q}_n^2$。换言之，2 层序和 2-QL 序是同一类因果集。

Causal Set Partition Function

因果集配分函数

Approaches to CST dynamics include sequential growth models [13] and continuum inspired dynamics using a CST partition function, also called as the path-sum. Here we will discuss the latter. For the sequential growth models, readers can refer to [9] and references therein.

CST 动力学的研究方向包括序增长模型 [13]，以及使用 CST 配分函数 (也称为路径和) 的受连续体启发的动力学。本文我们将讨论后者。关于序增长模型，读者可参考 [9] 及其中的相关文献。

The CST partition function (path-sum) over a sample space Ω is given by

样本空间 Ω 上的 CST 配分函数 (路径和) 由下式给出

$$\mathcal{Z} = \sum_{C \in \Omega} \exp\left(\frac{i}{\hbar}\mathcal{S}(C)\right) \tag{2}$$

where $\mathcal{S}(C)$ denotes a choice of causal set action. The choice of sample space Ω depends on the problem at hand, and in this chapter we will work with $\Omega = \Omega_n$, which is the sample space of n-element labeled causal sets. The path-sum Eq. (2) can be split as a sum over non-unique mutually disjoint subsets $\Omega^{(1)}, \Omega^{(2)}, \ldots$ of Ω, i.e.,

其中 $\mathcal{S}(C)$ 表示选定的因果集作用量。样本空间 Ω 的选择取决于当前研究问题，本章我们将在 $\Omega = \Omega_n$ 中开展研究，$\Omega = \Omega_n$ 是 n 个元素的标号因果集的样本空间。路径和式 (2) 可以拆分为对 Ω 的非唯一互斥子集 $\Omega^{(1)}, \Omega^{(2)}, \ldots$ 求和，即

$$\mathcal{Z} = \mathcal{Z}^{(1)} + \mathcal{Z}^{(2)} + \ldots, \tag{3}$$

where

其中

$$\mathcal{Z}^{(m)} = \sum_{C \in \Omega^{(m)}} \exp\left(\frac{i}{\hbar}\mathcal{S}(C)\right) \tag{4}$$

This split allows us to inspect the contribution to the path-sum from different classes of causal sets.

这种拆分可以让我们分别分析不同类别因果集对路径和的贡献。

A natural choice of the causal set action, $\mathcal{S}(C)$ is the Benincasa-Dowker-Glaser (BDG) action,

因果集作用量 $\mathcal{S}(C)$ 的一个自然选择是 Benincasa-Dowker-Glaser(BDG) 作用量,

$$\frac{1}{\hbar}\mathcal{S}_{BDG}^{(d)}(C) \equiv \mu(d)\left(n + \sum_{j=0}^{j_{\max}(d)} \lambda_j(d) N_j\right), \tag{5}$$

where N_j is the number of j -element order intervals in C and $\mu(d), \lambda_j(d)$ and $j_{\max}(d)$ are dimension dependent constants (see [14-16] for details). In particular for $d = 4$ we have

其中 N_j 是 C 中 j 元序区间的数量，$\mu(d), \lambda_j(d)$ 和 $j_{\max}(d)$ 是依赖于维度的常数 (细节见 [14-16])。特别地，对于 $d = 4$ 我们有

$$\frac{1}{\hbar}\mathcal{S}_{BDG}^{(4)}(C) = \left(\frac{l}{l_p}\right)^2 (n - N_0 + 9N_1 - 16N_2 + 8N_3), \tag{6}$$

where l_p is the Planck length, and l is the discreteness scale determined by the sprinkling density.

其中 l_p 是普朗克长度，l 是由撒播密度确定的离散标度。

In a sample space of causal sets with large number of elements, the expectation value of $\mathcal{S}(C)$ over different Poisson sprinkling gives the Einstein-Hilbert action, up to boundary terms [17-19].

在元素数量很大的因果集样本空间中，不同泊松散布下 $\mathcal{S}(C)$ 的期望在边界项修正内等于爱因斯坦-希尔伯特作用量 [17-19]。

The BDG action is defined via a top-down approach, i.e., instead of defining the coefficients using order theoretic arguments within CST, they are determined by comparing the average of BDG action over different sprinklings with a known continuum action. They are both dimension-dependent and apparently arbitrary.

BDG 作用量通过自顶向下方法定义: 即不对系数使用 CST 内的序理论论证来定义，而是通过将不同撒播下 BDG 作用量的平均值与已知连续体作用量对比来确定系数。这些系数既依赖维度，也明显带有任意性。

A simpler choice of action, called the link action, is proposed in [6, 20], which can be considered as a more natural choice from an order theoretic perspective. The link action depends only on the number of elements n and the number of links N_0

文献 [6, 20] 提出了一种更简单的作用量选择，称为链接作用量；从序理论的角度看，它是更自然的选择。链接作用量仅依赖元素数量 n 和链接数量 N_0

$$\frac{1}{\hbar}\mathcal{S}_L(C) \equiv \mu(n + \lambda_0 N_0) \tag{7}$$

which can also be obtained from the BDG action by putting $\lambda_j = 0, \forall j > 0$. The continuum limit of this action is still unknown.

它也可以通过对 BDG 作用量代入 $\lambda_j = 0, \forall j > 0$ 得到。该作用量的连续体极限目前仍未知。

Suppression of Layered Causal Sets

分层因果集的抑制

We now have all the ingredients needed to study the suppression of layered causal sets. In this section, we discuss the suppression of k -QL orders in Ω_n by the link action.

我们现已掌握研究分层因果集抑制所需的全部基础条件。在本节中，我们将讨论连接作用对 Ω_n 中 k -QL 序的抑制。

Counting n -Element k -QL Orders

计数 n 元 k -QL 序

Definition 11-2 makes direct counting of k -QL orders difficult. We therefore count them indirectly using another class of layered orders, which is defined in [6] and is called as the pseudo-quasi-layer orders that satisfy less stringent conditions than k -QL orders. We determine the upper and the lower bounds on the number of n -element k -QL orders in terms of that of n -element k -PQL orders and show that the log of the upper and the lower bounds are identical up to the leading order in n .

定义 11-2 使得直接计数 k -QL 序非常困难，因此我们通过另一类分层序间接计数，这类序定义于文献 [6]，称为伪拟分层序，满足比 k -QL 序更宽松的条件。我们借助 n 元 k -PQL 序的数量，确定 n 元 k -QL 序数量的上下界，并证明在 n 下，上下界的对数首项是相同的。

Definition 12. In a k -pseudo-quasi-layer (k -PQL) order $C \in \Omega_n$ it is possible to assign a k -layer-map ϑ , such that Definition 11-1 is satisfied but not necessarily Definition 11-2. We will refer to this k -layer-map assignment as pseudo-quasi-layers (PQL).

定义 12. 在 k 元伪拟分层 (k -PQL) 序 $C \in \Omega_n$ 中，可以分配一个 k 层映射 ϑ ，满足定义 11-1，但不一定满足定义 11-2。我们将这种 k 层映射分配称为伪拟分层 (PQL)。

Let $\mathcal{P}_n^k$ denote the set of k -PQL orders. We build up k -PQL orders by assigning k -layer map to n elements and add in relations that satisfy Definition 12. Different k -PQL assignment to n elements can lead to the same causal set in Ω_n , leading to over-counting. The k -PQL order is therefore not a subset of Ω_n .

设 $\mathcal{P}_n^k$ 为所有 k -PQL 序构成的集合。我们通过给 n 个元素分配 k 层映射，添加满足定义 12 的关系来构造 k -PQL 序。给 n 元素分配的不同 k -PQL，在 Ω_n 中可能对应同一个因果集，导致重复计数。因此 k -PQL 序不是 Ω_n 的子集。

Let $\mathcal{P}_{\vec{q},n}$ denote a k -PQL order with n elements and filling fraction $\vec{q} = (q_1, q_2, \ldots, q_k)$, such that the first PQL contains $q_1 n$ elements, second PQL contains $q_2 n$ elements, and so on.

设 $\mathcal{P}_{\vec{q},n}$ 为一个含 n 个元素、填充率为 $\vec{q} = (q_1, q_2, \ldots, q_k)$ 的 k -PQL 序，满足第一个 PQL 层含 $q_1 n$ 个元素，第二个 PQL 层含 $q_2 n$ 个元素，依此类推。

Definition 13. A PQL assignment is said to be naturally labeled if the first PQL is $\{e_1, \dots, e_{q_1 n}\}$, the second PQL is $\{e_{q_1 n+1}, \dots, e_{(q_1+q_2)n}\}$ and so on, and therefore is unique. We denote the set of naturally labeled k -PQL orders with filling fraction $\vec{q}$ by $\mathcal{P}^*_{\vec{q},n} \subset \mathcal{P}^k_n$.

定义 13. 如果第一个 PQL 层是 $\{e_1, \dots, e_{q_1 n}\}$，第二个 PQL 层是 $\{e_{q_1 n+1}, \dots, e_{(q_1+q_2)n}\}$，依此类推，则称该 PQL 分配是自然标号的，且这种分配是唯一的。我们将填充率为 $\vec{q}$ 的所有自然标号 k -PQL 序构成的集合记为 $\mathcal{P}^*_{\vec{q},n} \subset \mathcal{P}^k_n$ 。

Let $\mathcal{P}^*_{\vec{q},p,n}$ denote a set of all naturally labeled k -PQL orders with n elements, filling fraction $\vec{q}$ and pn^2 links and $\mathcal{Q}^k_{p,n}$ denote a set of all k -QL orders with n elements and pn^2 links. It is shown in [6] that there exists a one-to-one map from $\mathcal{P}^*_{\vec{q},p,n}$ to $\mathcal{Q}^k_{p,n}$, and therefore

设 $\mathcal{P}^*_{\vec{q},p,n}$ 为所有含 n 个元素、填充率为 $\vec{q}$ 、有 pn^2 条连接的自然标号 k -PQL 序构成的集合，$\mathcal{Q}^k_{p,n}$ 为所有含 n 个元素、有 pn^2 条连接的 k -QL 序构成的集合。文献 [6] 已经证明，存在从 $\mathcal{P}^*_{\vec{q},p,n}$ 到 $\mathcal{Q}^k_{p,n}$ 的双射，因此

$$\left|\mathcal{P}^*_{\vec{q},p,n}\right| < \left|\mathcal{Q}^k_{p,n}\right| \tag{8}$$

Note that Eq. (8) is true for any choice of $\vec{q}$ including the one that maximizes $\left|\mathcal{P}^*_{\vec{q},p,n}\right|$. Let $\vec{q}_0$ be one of the filling fractions for which $\left|\mathcal{P}^*_{\vec{q}_0,p,n}\right|$ is maximum over $\vec{q}$ ($\vec{q}_0 = (1/2, 1/2)$ and $\vec{q}_0 = (1/4-x, 1/2, 1/4+x)$ are the choices for $\vec{q}_0$.). due to the over-counting in k -PQL orders,

注意，式 (8) 对 $\vec{q}$ 的任意选取都成立，包括使 $\left|\mathcal{P}^*_{\vec{q},p,n}\right|$ 取最大值的选取。由于 k -PQL 序中存在重复计数，设 $\vec{q}_0$ 是使得 $\left|\mathcal{P}^*_{\vec{q}_0,p,n}\right|$ 在 $\vec{q}$($\vec{q}_0 = (1/2, 1/2)$ and $\vec{q}_0 = (1/4-x, 1/2, 1/4+x)$ are the choices for $\vec{q}_0$.). Also,) 上取得最大值的填充分数之一，

$$\left|\mathcal{Q}^k_{p,n}\right| < \left|\mathcal{P}^k_{p,n}\right| \tag{9}$$

where $\mathcal{P}^k_{p,n}$ denote a set of all k -PQL orders with pn^2 links.

其中 $\mathcal{P}^k_{p,n}$ 表示所有含 pn^2 条链接的 k -PQL 序构成的集合。

Using Eqn (8) with $\vec{q} = \vec{q}_0$,(9) and the fact that

将式 (8) 与 $\vec{q} = \vec{q}_0$ 、式 (9) 结合，再结合以下事实:

$$\left|\mathcal{P}^k_{p,n}\right| = \sum_{\vec{q}} m(\vec{q}) \left|\mathcal{P}^*_{\vec{q},p,n}\right| \le \sum_{\vec{q}} m(\vec{q}) \left|\mathcal{P}^*_{\vec{q}_0,p,n}\right| = k^n \left|\mathcal{P}^*_{\vec{q}_0,p,n}\right|, \tag{10}$$

where

其中

$$m\left(\vec{q}\right)=\frac{n!}{(q_1n)!\,(q_2n)!\ldots(q_kn)!} \tag{11}$$

is the number of ways of filling the PQL's with a given filling fraction $\vec{q}$, we find that

是给定填充分数 $\vec{q}$ 时填充 PQL 的方法数，我们得到:

$$\left|\mathcal{P}^{*}_{\vec{q}_0,p,n}\right|<\left|\mathcal{Q}^{k}_{p,n}\right|<k^{n}\left|\mathcal{P}^{*}_{\vec{q}_0,p,n}\right|. \tag{12}$$

$$\left|\mathcal{P}^{*}_{\vec{q}_0,p,n}\right|=\binom{\alpha\left(\vec{q}_0\right)n^2}{pn^2}, \tag{13}$$

where $\alpha\left(\vec{q}\right)n^2$ is the maximum number of links possible in a k -PQL order with filling fraction $\vec{q}$. Using Stirling’ s approximation, to the leading order in n ,

其中 $\alpha\left(\vec{q}\right)n^2$ 是填充分数为 $\vec{q}$ 的 k -PQL 序中最多可拥有的链接数。利用斯特林近似，在 n 的领头阶下，

$$\ln\left|\mathcal{P}^{*}_{\vec{q}_0,p,n}\right|=\alpha\left(\vec{q}_0\right)n^2h\left(\tilde{p}\right)+o\left(n^2\right), \tag{14}$$

where $0<\tilde{p}=\alpha(\vec{q}_0)^{-1}p<1$ and

其中 $0<\tilde{p}=\alpha(\vec{q}_0)^{-1}p<1$ 且

$$h\left(\tilde{p}\right)=-\tilde{p}\ln\left(\tilde{p}\right)-\left(1-\tilde{p}\right)\ln\left(1-\tilde{p}\right). \tag{15}$$

Also up to the order of n^2 , the explicit counting of $\mathcal{P}^{k}_{p,n}$ in Eq. (10), with the extra factor of k^n leads to the same type of expression as that of the bilayer order in Eq. (14),

同样在 n^2 阶范围内，式 (10) 中对 $\mathcal{P}^{k}_{p,n}$ 的显式计数在额外因子 k^n 的作用下，得到的结果与式 (14) 中双层序的表达式形式一致，

$$\ln\left(k^n\left|\mathcal{P}^{*}_{\vec{q}_0,p,n}\right|\right)=n\ln k+\ln\left|\mathcal{P}^{*}_{\vec{q}_0,p,n}\right|=\alpha\left(\vec{q}_0\right)n^2h\left(\tilde{p}\right)+o\left(n^2\right). \tag{16}$$

Equations (12),(14) and (16) suggest that to the leading order in n ,

式 (12)、(14) 和 (16) 表明，在 n 的领头阶下，

$$\ln\left|\mathcal{Q}^{k}_{p,n}\right|=\ln\left|\mathcal{P}^{*}_{\vec{q}_0,p,n}\right|+o\left(n^2\right)=\alpha\left(\vec{q}_0\right)n^2h\left(\tilde{p}\right)+o\left(n^2\right). \tag{17}$$

Calculation of the Path-Sum for n -Element k -QL Orders

n 元 k -QL 序的路径和计算

The contribution of n -element k -QL orders to the causal set path-sum can be written as

n 元 k -QL 序对因果集路径和的贡献可写为

$$\mathcal{Z}|_{Q_n^k} = \int_0^1 d\tilde{p}\,|Q_{p,n}^k| \exp\left(\frac{i}{\hbar}\mathcal{S}(C)\right). \tag{18}$$

Here it is assumed that n is large for which $\tilde{p}$ can be treated like a continuous variable.

此处假设 n 很大，此时 $\tilde{p}$ 可被当作连续变量处理

Note that the Kleitman-Rothschild result implies that if $\mathcal{S}(C) = 0$, the KR orders dominate the path-sum. The choice of $\mathcal{S}(C)$ is therefore crucial in taming the contribution of the KR orders. Here we compute the path-sum with the link action and show that this choice of action suppresses the contribution of the KR orders.

需要注意，Kleitman-Rothschild 结果表明: 当 $\mathcal{S}(C) = 0$ 时，KR 序在路径和中占主导。因此 $\mathcal{S}(C)$ 的选取对于约束 KR 序的贡献至关重要。本文我们计算带链接作用量的路径和，证明这种作用量的选择会抑制 KR 序的贡献。

For an n -element k -QL order with pn^2 links, the link action (Eq. (7)) takes the form

对于一个包含 pn^2 条链接的 n 元 k -QL 序，链接作用量 (式 (7)) 形式为

$$\mathcal{S}_L(C) = \mu n + \mu\lambda_0 pn^2 = \mu n + \mu\lambda_0\alpha\left(\vec{q}_0\right)\tilde{p}n^2, \tag{19}$$

and therefore

因此可得

$$\begin{aligned}\mathcal{Z}|_{Q_n^k} &= \int_0^1 d\tilde{p}\exp\left(n^2\alpha\left(\vec{q}_0\right)\left(i\mu\lambda_0\tilde{p} + h(\tilde{p})\right) + o\left(n^2\right)\right)\\ &= \int_0^1 d\tilde{p}\exp\left(n^2\alpha\left(\vec{q}_0\right)E(\tilde{p}) + o\left(n^2\right)\right),\end{aligned} \tag{20}$$

where

其中

$$E(\tilde{p}) = i\mu\lambda_0\tilde{p} + h(\tilde{p}). \tag{21}$$

We will solve the integral in Eq. (20) using the method of steepest descent as is done in [5]. We first find the saddle point:

我们将沿用文献 [5] 的最陡下降法求解式 (20) 中的积分，首先找到鞍点:

$$E'(\widetilde{p}) = i\mu\lambda_0 - \ln\widetilde{p} + \ln(1-\widetilde{p}) = 0, \tag{22}$$

$$\Rightarrow \widetilde{p}_0 = \frac{e^{i\mu\lambda_0}}{1+e^{i\mu\lambda_0}} = \frac{e^{i\mu\lambda_0/2}}{2\cos(\mu\lambda_0/2)} = \frac{1}{2}\left(1+i\tan\left(\frac{\mu\lambda_0}{2}\right)\right). \tag{23}$$

At $\widetilde{p} = \widetilde{p}_0$, the second derivative of E is

当 $\widetilde{p} = \widetilde{p}_0$ 时，E 的二阶导数为

$$E''(\widetilde{p}_0) = -\frac{1}{\widetilde{p}_0} - \frac{1}{1-\widetilde{p}_0} = -4\cos^2\left(\frac{\mu\lambda_0}{2}\right). \tag{24}$$

Since $E''(\widetilde{p}_0) < 0$, the steepest descent is along the direction in which $\widetilde{p} - \widetilde{p}_0$ is real. At the saddle point

由于 $E''(\widetilde{p}_0) < 0$ ，最陡下降沿 $\widetilde{p} - \widetilde{p}_0$ 为实数的方向进行，在鞍点处

$$h(\widetilde{p}_0) = \frac{\mu\lambda_0}{2}\tan\left(\frac{\mu\lambda_0}{2}\right) + \ln\left(2\cos\left(\frac{\mu\lambda_0}{2}\right)\right) \tag{25}$$

$$E(\widetilde{p}_0) = i\frac{\mu\lambda_0}{2} + \ln\left(2\cos\left(\frac{\mu\lambda_0}{2}\right)\right). \tag{26}$$

Saddle-point approximation of the integral in Eq. (20) around $\widetilde{p} = \widetilde{p}_0$ is

式 (20) 中积分在 $\widetilde{p} = \widetilde{p}_0$ 附近的鞍点近似为

$$\mathcal{Z}|_{\mathcal{Q}_n^k} \approx \int_0^1 d\widetilde{p}\exp\left(n^2\alpha(\vec{q}_0)\left(E(\widetilde{p}_0) + \frac{E''(\widetilde{p}_0)}{2}(\widetilde{p}-\widetilde{p}_0)^2\right) + o\left(n^2\right)\right). \tag{27}$$

In the limit of large n , the integral above is a Gaussian integral, which when solved gives

在大 n 极限下，上述积分是高斯积分，求解后可得

$$\mathcal{Z}|_{\mathcal{Q}_n^k} \approx \frac{1}{n}\sqrt{\frac{2\pi}{\alpha(\vec{q}_0)\,|E''(p_0)|}}\exp\left(n^2\alpha(\vec{q}_0)\,E(\widetilde{p}_0) + o\left(n^2\right)\right). \tag{28}$$

Contribution of $\mathcal{Q}_n^k$ to the path-sum is therefore suppressed if $\mathrm{Re}(E(\widetilde{p}_0)) < 0$, which is true when $|\cos(\mu\lambda_0/2)| < 1/2$. Since the saddle point lies outside the real axis of the complex plane, one needs to deform the contour in the integral of Eq. (27) to make it pass through the saddle point, as shown in Fig. 8. We therefore have to check that the rest of the contour does not contribute significantly to the integral.

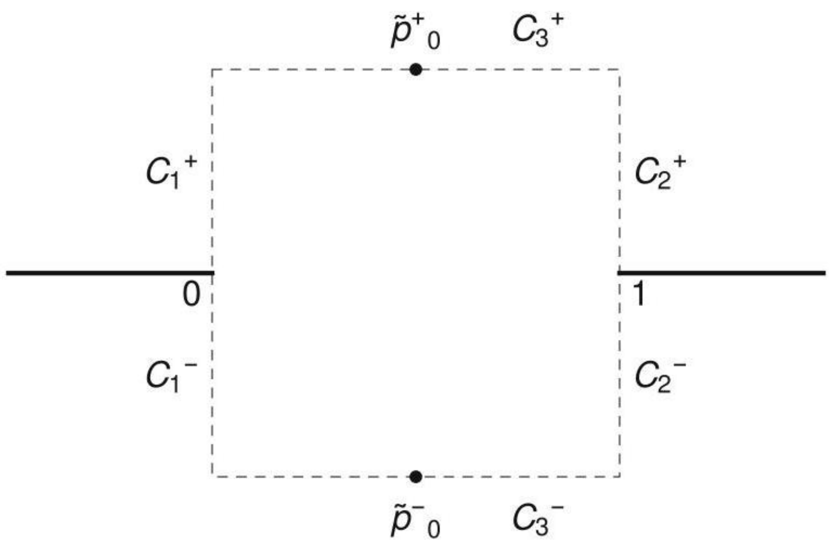
因此当 $\mathrm{Re}(E(\widetilde{p}_0)) < 0$ 时，$\mathcal{Q}_n^k$ 对路径和的贡献被抑制，这在 $|\cos(\mu\lambda_0/2)| < 1/2$ 时成立。由于鞍点位于复平面实轴之外，需要变形式 (27) 积分的围道使其经过鞍点，如图 8 所示。因此我们需要验证围道其余部分对积分的贡献不显著。

If $\mathrm{Im}(\widetilde{p}_0) = \tan(\mu\lambda_0/2)/2 > 0$, then we deform the contour along the path $C_1^+ \cup C_3^+ \cup C_2^+$ in the upper half complex plain in order to make it pass through $\widetilde{p}_0 = \widetilde{p}_0^+$ as shown in Fig. 8. If $\tan(\mu\lambda_0/2) < 0$, then we deform the contour along the path $C_1^- \cup C_3^- \cup C_2^-$ in the lower half complex plain in order to make it pass through $\widetilde{p}_0 = \widetilde{p}_0^-$.

当 $\mathrm{Im}(\widetilde{p}_0) = \tan(\mu\lambda_0/2)/2 > 0$ 时，我们沿复上半平面的路径 $C_1^+ \cup C_3^+ \cup C_2^+$ 变形围道，使其经过如图 8 所示的 $\widetilde{p}_0 = \widetilde{p}_0^+$；当 $\tan(\mu\lambda_0/2) < 0$ 时，我们沿复下半平面的路径 $C_1^- \cup C_3^- \cup C_2^-$ 变形围道，使其经过 $\widetilde{p}_0 = \widetilde{p}_0^-$ 。

Fig. 8 Contour from $\widetilde{p} = 0$ to $\widetilde{p} = 1$ (dashed line) passing through the saddle point $\widetilde{p}_0$, which is shown as $\widetilde{p}_0^+$ if $\mathrm{Im}(\widetilde{p}_0) > 0$ and $\widetilde{p}_0^-$ if $\mathrm{Im}(\widetilde{p}_0) < 0$

图 8 从 $\widetilde{p} = 0$ 到 $\widetilde{p} = 1$ (虚线) 经过鞍点 $\widetilde{p}_0$ 的围道，当 $\mathrm{Im}(\widetilde{p}_0) > 0$ 时鞍点标记为 $\widetilde{p}_0^+$ ，当 $\mathrm{Im}(\widetilde{p}_0) < 0$ 时标记为 $\widetilde{p}_0^-$



Case 1: $\tan(\mu\lambda_0/2) > 0$

情形 1: $\tan(\mu\lambda_0/2) > 0$

The path from $\widetilde{p} = 0$ to $\widetilde{p} = 1$ consists of C_1^+, C_2^+ and C_3^+ . On C_1^+ , let $\widetilde{p} = iw$, where $0 < w < \frac{1}{2}\tan(\mu\lambda_0/2)$.

从 $\widetilde{p} = 0$ 到 $\widetilde{p} = 1$ 的路径由 C_1^+, C_2^+ 和 C_3^+ 构成。令 $\widetilde{p} = iw$ 定义在 C_1^+ 上，其中 $0 < w < \frac{1}{2}\tan(\mu\lambda_0/2)$ 。

$$\ln\widetilde{p} = \ln w + i\frac{\pi}{2},\ \ln(1-\widetilde{p}) = \ln\sqrt{1+w^2} - i\tan w \tag{29}$$

$h(\widetilde{p})$ and $E(\widetilde{p})$ for $\widetilde{p} = iw$ takes the form

对应 $\widetilde{p} = iw$ 的 $h(\widetilde{p})$ 与 $E(\widetilde{p})$ 形式为

$$h\left(iw\right)=\frac{\pi}{2}w+w\tan^{-1}w-\ln\sqrt{1+w^{2}}+i\,\mathrm{Im}\left(h\left(iw\right)\right),\tag{30}$$

$$E\left(iw\right)=\frac{\pi}{2}w+w\tan^{-1}w-\mu\lambda_{0}w-\ln\sqrt{1+w^{2}}+i\,\mathrm{Im}\left(E\left(iw\right)\right).\tag{31}$$

On C_2^+ , let $\widetilde{p}=1+iw$, where $0<w<\tan\left(\mu\lambda_0/2\right)$.

令 $\widetilde{p}=1+iw$ 定义在 C_2^+ 上，其中 $0<w<\tan\left(\mu\lambda_0/2\right)$ 。

$$\ln\widetilde{p}=\ln\sqrt{1+w^{2}}+i\tan w,\ \ln\left(1-\widetilde{p}\right)=\ln w-i\frac{\pi}{2}.\tag{32}$$

It can easily be checked that $\mathrm{Re}\left(h\left(iw\right)\right)=\mathrm{Re}\left(h\left(1+iw\right)\right)$ and $\mathrm{Re}\left(E\left(iw\right)\right)=\mathrm{Re}\left(E\left(1+iw\right)\right)$ given by Eq. (31). If $\mu\lambda_0<0$ then $\mathrm{Re}\left(E\left(iw\right)\right)>0$, which means that the upper contour C_1^+ and C_2^+ will contribute significantly to the path-sum, which then won' t be suppressed, and hence, this contour is ruled out. If $\mu\lambda_0>\pi$, then $\mathrm{Re}\left(E\left(iw\right)\right)<0$ and the contribution of C_1^+ and C_2^+ to the path-sum is suppressed.

不难验证，$\mathrm{Re}\left(h\left(iw\right)\right)=\mathrm{Re}\left(h\left(1+iw\right)\right)$ 和 $\mathrm{Re}\left(E\left(iw\right)\right)=\mathrm{Re}\left(E\left(1+iw\right)\right)$ 由式 (31) 给出。若 $\mu\lambda_0<0$ ，则 $\mathrm{Re}\left(E\left(iw\right)\right)>0$ ，这意味着上围道 C_1^+ 和 C_2^+ 会对路径和产生显著贡献，路径和不会被抑制，因此该围道被排除。若 $\mu\lambda_0>\pi$ ，则 $\mathrm{Re}\left(E\left(iw\right)\right)<0$ ，C_1^+ 和 C_2^+ 对路径和的贡献被抑制。

If $0<\mu\lambda_0<\pi$, then the condition that $\left|\cos\left(\mu\lambda_0/2\right)\right|<1/2$ limits us to the range $2\pi/3<\mu\lambda_0<\pi$. We look for the maximum value of $E\left(iw\right)$ over w for a given $\mu\lambda_0$ in this range and check if it goes above zero. At $w=0,\mathrm{Re}\left(E\left(iw\right)\right)=0$ and

若 $0<\mu\lambda_0<\pi$ ，则 $\left|\cos\left(\mu\lambda_0/2\right)\right|<1/2$ 的条件将我们限制在区间 $2\pi/3<\mu\lambda_0<\pi$ 内。我们在该区间内对任意给定的 $\mu\lambda_0$ ，寻找 $E\left(iw\right)$ 在 w 上的最大值，并检查其是否大于零。在 $w=0,\mathrm{Re}\left(E\left(iw\right)\right)=0$ 处且

$$\frac{d\,\mathrm{Re}\left(E\left(iw\right)\right)}{dw}=\frac{\pi}{2}+\tan^{-1}w-\mu\lambda_{0},\tag{33}$$

which is less than zero for small w , which means that $\mathrm{Re}\left(E\left(iw\right)\right)$ decreases with the increase in w . It attains a minimum when

当 w 很小时，该式小于零，意味着 $\mathrm{Re}\left(E\left(iw\right)\right)$ 随 w 的增大而减小。它在以下位置取得最小值

$$\frac{\pi}{2}+\tan^{-1}w-\mu\lambda_{0}=0\Rightarrow w=-\cot\left(\mu\lambda_{0}\right)=\frac{1}{2}\left(\tan\left(\frac{\mu\lambda_{0}}{2}\right)-\cot\left(\frac{\mu\lambda_{0}}{2}\right)\right).$$

(34)

So the minima lies in the range $0<w<\frac{1}{2}\tan\left(\mu\lambda_0/2\right)$, the maximum therefore may lie at either bound of w , but one can check that at $w=\frac{1}{2}\tan\left(\mu\lambda_0/2\right),\mathrm{Re}\left(E\left(iw\right)\right)<0$ for any value of $\mu\lambda_0$ in the range $2\pi/3<\mu\lambda_0<\pi$.

因此最小值位于区间 $0<w<\frac{1}{2}\tan\left(\mu\lambda_0/2\right)$ 内，最大值可能出现在 w 的任意一个端点上，但可以验证，对于区间 $2\pi/3<\mu\lambda_0<\pi$ 内任意 $\mu\lambda_0$ 的值，在 $w=\frac{1}{2}\tan\left(\mu\lambda_0/2\right),\mathrm{Re}\left(E\left(iw\right)\right)<$ 处都满足值小于零。

Case 2: $\tan(\mu\lambda_0/2) < 0$

情况 2: $\tan(\mu\lambda_0/2) < 0$

The path from $\tilde{p} = 0$ to $\tilde{p} = 1$ consists of C_1^-, C_2^- and C_3^-. On C_1^-, let $\tilde{p} = -iw$, where $0 < w < \frac{1}{2}|\tan(\mu\lambda_0/2)|$.

从 $\tilde{p} = 0$ 到 $\tilde{p} = 1$ 的路径由 C_1^-, C_2^- 和 C_3^- 构成。令 $\tilde{p} = -iw$ 作用于 C_1^-，其中 $0 < w < \frac{1}{2}|\tan(\mu\lambda_0/2)|$ 。

$$\ln\tilde{p} = \ln w - i\frac{\pi}{2},\ \ln(1-\tilde{p}) = \ln\sqrt{1+w^2} + i\tan w \tag{35}$$

$h(\tilde{p})$ and $E(\tilde{p})$ for $\tilde{p} = -iw$ takes the form

$\tilde{p} = -iw$ 对应的 $h(\tilde{p})$ 和 $E(\tilde{p})$ 形式如下

$$h(-iw) = \frac{\pi}{2}w + w\tan^{-1}w - \ln\sqrt{1+w^2} + i\,\mathrm{Im}(h(-iw)), \tag{36}$$

$$E(-iw) = \frac{\pi}{2}w + w\tan^{-1}w + \mu\lambda_0 w - \ln\sqrt{1+w^2} + i\,\mathrm{Im}(E(-iw)). \tag{37}$$

On C_2^-, let $\tilde{p} = 1 - iw$, where $0 < w < \frac{1}{2}|\tan(\mu\lambda_0/2)|$. As in the previous case, it can easily be checked that $\mathrm{Re}(h(-iw)) = \mathrm{Re}(h(1-iw))$ and $\mathrm{Re}(E(-iw)) = \mathrm{Re}(E(1-iw))$ given by Eq. (37).

令 $\tilde{p} = 1 - iw$ 作用于 C_2^-，其中 $0 < w < \frac{1}{2}|\tan(\mu\lambda_0/2)|$ 。与前述情形相同，不难验证 $\mathrm{Re}(h(-iw)) = \mathrm{Re}(h(1-iw))$ 和 $\mathrm{Re}(E(-iw)) = \mathrm{Re}(E(1-iw))$ 由式 (37) 给出。

An analysis similar to that of the case of $\tan(\mu\lambda_0/2) > 0$ shows that in this case the contribution of the contours C_1^- and C_2^- to the path-sum is suppressed for if $\mu\lambda_0 < 0$

对 $\tan(\mu\lambda_0/2) > 0$ 情形的类似分析表明，若 $\mu\lambda_0 < 0$ 成立，则围道 C_1^- 和 C_2^- 对路径和的贡献是被压制的

Therefore, the contribution of k-QL orders to the causal set path-sum is exponentially suppressed if the parameters of the link action $\mu\lambda_0$ satisfy the following

因此，若连接作用量的参数 $\mu\lambda_0$ 满足下述条件，则 k 阶 k-QL 序对因果集路径和的贡献是指数压制的

$$\mu\lambda_0 > 0,\ \left|\cos\left(\frac{\mu\lambda_0}{2}\right)\right| < \frac{1}{2},\ \tan\left(\frac{\mu\lambda_0}{2}\right) > 0 \tag{38}$$

$$\Rightarrow \mu\lambda_0 > 0,\ \tan\left(\frac{\mu\lambda_0}{2}\right) > \sqrt{3} \tag{39}$$

or

$$\mu\lambda_0 < 0,\ \left|\cos\left(\frac{\mu\lambda_0}{2}\right)\right| < \frac{1}{2},\ \tan\left(\frac{\mu\lambda_0}{2}\right) < 0 \tag{40}$$

$$\Rightarrow \mu\lambda_0 < 0,\ \tan\left(\frac{\mu\lambda_0}{2}\right) < -\sqrt{3} \tag{41}$$

This analysis is taken a step further in [5] by considering higher-order terms in the saddle point approximation leading to a more tighter bound on the parameters given by

> 文献 [5] 将该分析更进一步，考虑了鞍点近似中的高阶项，得到了对参数更紧的界，如下所示

$$\mu\lambda_0 > 0,\ \tan\left(\frac{\mu\lambda_0}{2}\right) > \sqrt{\frac{27}{4}e^{-1/2} - 1} \tag{42}$$

or

$$\mu\lambda_0 < 0,\ \tan\left(\frac{\mu\lambda_0}{2}\right) < -\sqrt{\frac{27}{4}e^{-1/2} - 1}. \tag{43}$$

Further, a combinatorial argument is presented in [7] for three-layered orders, and generalized to all layers in [8], to show that the N_j terms for $j > 0$ in the BDG action contribute only to sub-leading orders in n , and to the leading order in n the link action dominates the BDG action. Hence, the results discussed here imply that all layered causal sets are suppressed in the full path-sum with the full BDG action, irrespective of the dimension of the BDG action.

> 此外，文献 [7] 针对分层序给出了组合论证，文献 [8] 将其推广到任意层数，证明了 BDG 作用量中对应 $j > 0$ 的 N_j 项仅对 n 中的次领头阶有贡献，在 n 的领头阶中连接作用量占 BDG 作用量的主导地位。因此本文讨论的结果表明，无论 BDG 作用量的维数如何，全路径和中所有分层因果集都被压制。

Conclusions

结论

We reviewed recent developments toward understanding the emergence of manifold-like behaviour of spacetime in the causal set theory. We started the chapter with a brief introduction to causal sets followed by a discussion on non-manifold-like causal sets. We then discussed the causal set actions and the path-sum. We saw that a particular class of non-manifold-like causal sets called here as the quasi-layer (QL) causal sets are suppressed in the causal set path-sum by the link action, for a suitable choice of action parameters, in the limit of large number of spacetime elements. These QL orders consist of the Kleitman-Rothschild (KR) orders, which dominates the collection of all n -element labelled causal sets Ω_n with large n . Further the analysis of [7] suggests that the KR orders are also suppressed by the BDG action (which, for a manifold-like causal set reduces to the Einstein-Hilbert action in the classical limit) as the BDG action is not very different from the link action for the KR orders in large n limit. The study of [7] was generalized in [8] for layered causal sets with more than three layers.

我们综述了因果集理论中理解类流形时空涌现的最新进展。本章首先简要介绍了因果集，随后讨论了非类流形因果集，接着探讨了因果集作用量与路径和。我们发现，在合适选取作用量参数、时空元素数量很大的极限下，本文称为准层 (QL) 因果集的这一类非类流形因果集会被连接作用量在因果集路径和中压低。这些 QL 序包含 Kleitman-Rothschild(KR) 序，当 n 很大时，KR 序在所有 n 个元素的标记因果集 Ω_n 中占主导地位。此外，文献 [7] 的分析表明，KR 序也会被 BDG 作用量压低——对于类流形因果集，BDG 作用量在经典极限下退化为爱因斯坦-希尔伯特作用量；因为在 n 很大的极限下，KR 序的 BDG 作用量与连接作用量差异很小。文献 [7] 的研究在文献 [8] 中被推广到了三层以上的层状因果集。

We here, along with the references herein including the original KR paper [1], work with labeled orders, but as discussed in [21], the labeling introduces a factor of at most $n\,!$, which is sub-dominant to the entropic factor of 2^{n^2} . Hence, much of the analysis is expected to be carried over to the unlabeled case.

包括原始 KR 论文 [1] 在内，本文及此处引用的研究都在处理标记序，但正如文献 [21] 所讨论的，标记最多引入一个 $n\,!$ 因子，该因子对 2^{n^2} 的熵因子而言是次主导的，因此大部分分析预期都可以推广到未标记的情况。

In spite of the progress made toward understanding the emergence of continuum manifold-like behaviour of spacetime in causal set theory, there are several open questions that need to be answered. One of them is about the contribution of different types of manifold-like causal sets to the path-sum. If, for example, spacetimes of dimension $4 + D$, with D compact dimensions are the right continuum approximation of CST, one has to explain why causal sets that approximate spacetimes of other dimensions are suppressed. In order to do such analysis, one needs to characterize manifold-like causal sets, which is challenging and a subject for future investigations.

尽管在因果集理论中理解连续类流形时空的涌现已经取得了不少进展，仍有多个待解决的开放问题。其中之一是不同类型的类流形因果集对路径和的贡献问题。例如，如果维度为 $4 + D$ 、其中 D 个维数紧致的时空是因果集理论 (CST) 的正确连续近似，我们就需要解释为什么近似其他维度时空的因果集会被压低。要开展这类分析，首先需要表征类流形因果集，这一点颇具挑战性，也是未来研究的方向。

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References

参考文献

1. D.J. Kleitman, B.L. Rothschild, Asymptotic enumeration of partial orders on a finite set. Trans. Am. Math. Soc. 205, 205-220 (1975)

2. D. Dhar, Entropy and phase transitions in partially ordered sets. J. Math. Phys. 19, 1711-1713 (1978)

3. D. Dhar, Asymptotic enumeration of partially ordered sets. Pac. J. Math. 90, 299-305 (1980)

4. H.J. Prömel, A. Steger, A. Taraz, Phase transitions in the evolution of partial orders. J. Comb. Theory Ser. A 94, 230-275 (2001)

5. S. Loomis, S. Carlip, Suppression of non-manifold-like sets in the causal set path integral. Class. Quant. Grav. 35(2), 024002 (2018)

6. A. Mathur, A.A. Singh, S. Surya, Entropy and the link action in the causal set path-sum. Class. Quant. Grav. 38(4), 045017 (2021)

7. P. Carlip, S. Carlip, S. Surya, Path integral suppression of badly behaved causal sets. arXiv:2209.00327 (2022)

8. P. Carlip, S. Carlip, S. Surya, Paper in preparation

9. S. Surya, The causal set approach to quantum gravity. Living Rev. Rel. 22(1), 5 (2019)

10. R.M. Wald, General Relativity (Chicago University Press, Chicago, 1984)

11. S.W. Hawking, G.F.R. Ellis, The Large Scale Structure of Space-Time. Cambridge Monographs on Mathematical Physics (Cambridge University Press, New York, 1973)

12. G. Brightwell, H.J. Prömel, A. Steger, The average number of linear extensions of a partial order. J. Comb. Theory Ser. A 73, 193-206 (1996)

13. D.P. Rideout, R.D. Sorkin, A classical sequential growth dynamics for causal sets. Phys. Rev. D 61, 024002 (2000)

14. D.M. Benincasa, F. Dowker, The scalar curvature of a causal set. Phys. Rev. Lett. 104, 181301 (2010)

15. F. Dowker, L. Glaser, Causal set d'Alembertians for various dimensions. Class. Quant. Grav. 30, 195016 (2013)

16. L. Glaser, A closed form expression for the causal set d'Alembertian. Class. Quant. Grav. 31, 095007 (2014)

17. M. Buck, F. Dowker, I. Jubb, S. Surya, Boundary terms for causal sets. Class. Quant. Grav. 32(20), 205004 (2015)

18. F. Dowker, Boundary contributions in the causal set action. arXiv:2007.13206 (2020)

19. L. Machet, J. Wang, On the continuum limit of Benincasa-Dowker-Glaser causal set action. Class. Quant. Grav. 38(1), 015010 (2021)

20. A. Eichhorn, Towards coarse graining of discrete Lorentzian quantum gravity. Class. Quant. Grav. 35(4), 044001 (2018)

21. J. Henson, D.P. Rideout, R.D. Sorkin, S. Surya, Onset of the asymptotic regime for finite orders. arXiv:1504.05902 (2015)